

LINEAR HYPERBOLIC LACTATION CURVES

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ABSTRACT

The linear hyperbolic model $Y = \beta_0 + \beta_1 X + \beta_2 1/X$ and the gamma model $Y = Ax^b e^{-cx}$ were used to describe lactation curves of 553 Gyr dairy cows, considering the first three parturitions. The coefficient of determination for lactation stage means was 0.96 for both models. When the models were used individually for each cow in the herd, the linear hyperbolic model showed a slightly better fit than the gamma model. Due to the simplicity of its equation, the linear hyperbolic model can be used to advantage with respect to the gamma model for estimating total milk production of Gyr cows with incomplete records.

INTRODUCTION

Wood (1967) presented the following model

$$Y = Ax^b e^{-cx} \quad (1)$$

to describe lactations of individual animals or groups of animals. In this model, Y is mean daily milk yield during the Xth lactation week, and A, b and c are positive parameters that determine the shape of the lactation curve. In subsequent studies, Wood (1968, 1969, 1970a,b, 1972, 1976) discussed properties of the lactation curve and introduced a measurement of persistency of lactation in terms of b and c, and applied this model to a series of studies. Little has been done to describe lactation curves using new mathematical models. Most authors prefer to use the gamma curve which provides a good fit.

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A new and quite simple algebraic model was proposed by Bianchini (1984) to describe a single lactation and an estimate of total production. The model, denoted by the author as linear hyperbolic, has been used to describe the lactation of Gyr dairy cows and has the following equation:

$$Y = \beta_0 + \beta_1 X + \beta_2 / X \quad (2)$$

where Y is milk production at stage X , and β_0 , β_1 and β_2 are parameters that determine the shape of the curve. Model (2) may exhibit a peak for values of $X = \sqrt{\beta_2 / \beta_1}$ when $\beta_1 < 0$ and $\beta_2 < 0$.

The line $Y = \beta_0 + \beta_1 X$ is the oblique asymptote of the curve and its angular coefficient β_1 can be taken as the rate of decline (in kg) per day, thus supplying a measurement of the persistency of lactation for lactations having the same initial level. The parameter β_0 is the intercept of the asymptote of the curve with the Y axis.

Objectives of the present study were to demonstrate use of the linear hyperbolic equation in a Gyr dairy herd, to compare it with the gamma curve and to discuss some of its properties.

MATERIAL AND METHODS

Data used were 12,879 records of production of 553 cows with 1,084 lactations, considering only the first three lactations of a Gyr dairy herd belonging to Santa Ana da Serra farm, township of Cajuru, State of São Paulo, Brazil, from 1961 to 1979. Time periods within lactation were grouped into class intervals, denoted lactation stages, and are shown in Table I by order of lactation. The climate of this region is of the subtropical type, with dry winters. Rainfall distribution suggests definition of two calving seasons, a dry one from April to September, and a rainy one from October to March.

In model (1), the errors are assumed to be multiplicative and are transformed into additive by log transformation. The transformation used, $\log_e Y = \log_e A + b \log_e X - cX$ permits obtaining the estimators of $\log_e A$, b and $-c$, as a multiple linear regression of $\log_e Y$ on $\log_e X$ and S . Parameters of equation (2) were estimated by a multiple linear regression model of Y on X and $1/X$. To compare degree of fit of the two models, after application of the models to all the animals in the herd and analysis of residuals, the criterion used was frequency distribution of the coefficients of determination calculated by the following formula:

$$R^2 = \frac{\text{Sum of squares due to regression}}{\text{Total corrected sum of squares}}$$

Table I - Distribution of records by stage and order of lactation.

Stage	Days in milk	Lactation 1	Lactation 2	Lactation 3
1	0 - 20	270	257	165
2	21 - 30	197	142	95
3	31 - 40	198	149	94
4	41 - 50	168	151	92
5	51 - 60	184	140	94
6	61 - 70	183	133	81
7	71 - 80	165	162	105
8	81 - 100	355	266	173
9	101 - 120	366	291	181
10	121 - 150	513	411	259
11	151 - 180	512	403	255
12	181 - 210	501	402	257
13	211 - 240	489	377	249
14	241 - 270	458	365	230
15	271 - 300	420	326	197
16	301 - 330	354	270	162
17	331 - 370	290	223	117
Total		5605	4468	2806

The general mixed model described by Harvey (1977), used to determine which factors influenced the shape of the lactation curve, is given by the equation:

$$Y_{ijkl} = \mu + A_i + B_{ij} + F_k + e_{ijkl},$$

where Y_{ijkl} = trait to be studied, i.e. a set of 1084 β_i estimated for both equations, μ = overall mean, A_i = sire effect (random), B_{ij} = within sire cow effect (random), F_k = a set of fixed effects (lactation order, calving season, number of milkings), and e_{ijkl} = random error.

RESULTS AND DISCUSSION

Tables II and III show the equations of the linear hyperbolic and gamma curves obtained for the means of the lactation stages by order of lactation and number of milkings. The Tables show that both curves have $R^2 \geq 0.96$, thus providing good fits. Figures 1 and 2, which represent the 2nd lactation with two milkings and the 3rd

Table II - Equations for the linear hyperbolic curve obtained for mean data referring to three lactations.

Lactation	No. of milkings	Equation	Peak		R ²
			Days	Milk (kg)	
1st	2	Y = 8,852457 - 0.019380 X - 4.321233 1/X	14.97	8.27	0.96
1st	3	Y = 14.829935 - 0.024550 X - 35.903335 1/X	38.24	12.95	0.97
2nd	2	Y = 10.622636 - 0.017662 X + 13.770324 1/X	—	—	0.96
2nd	3	Y = 15.452589 - 0.026935 X - 8.975588 1/X	18.25	14.66	0.98
3rd	2	Y = 11.357765 - 0.020823 X + 9.875588 1/X	—	—	0.96
3rd	3	Y = 16.697980 - 0.031783 X - 7.345128 1/X	15.39	15.73	0.97

Table III - Equations for curves obtained for the mean data for three lactations.

Lactation	No. of milkings	Equation	Peak		R ²
			Days	Milk (kg)	
1st	2	Y = 7.825620 X ^{0.036010} e ^{-0.001840 X}	19.61	8.40	0.97
1st	3	Y = 7.888490 X ^{0.168728} e ^{-0.003458 X}	48.79	12.84	0.97
2nd	2	Y = 13.366688 X ^{-0.043835} e ^{-0.002132 X}	—	—	0.98
2nd	3	Y = 12.780787 X ^{0.061980} e ^{-0.002987 X}	20.74	14.49	0.99
3rd	2	Y = 13.124333 X ^{-0.020194} e ^{-0.002644 X}	—	—	0.96
3rd	3	Y = 14.669082 X ^{0.048084} e ^{-0.003230 X}	14.89	15.91	0.98

lactation with 3 milkings, respectively, show that the curves do not always exhibit growth from calving to peak, with peak production occurring at the beginning of lactation. This disagreed with observations by Wood (1967) on parameter *b* of the gamma lactation curve, which in these cases is negative. Among the 1084 lactations studied, we found that 391 had atypical gamma curves with $b < 0$. The phenomenon observed here seems to be a characteristic of Gyr cows, whose lactations do not always exhibit growth until the peak, with the shape of the curve being practically linear in these cases and very well explained by the linear hyperbolic equation.

Table IV shows the frequency distributions of the coefficient of determination calculated for all cows in the herd. The table shows a slightly better fit for

Figure 1 - Linear hyperbolic lactation curve (——).

$$Y = 10.622636 - 0.017662 X + 13.770324 1/X \quad R^2 = 0.96$$

Gamma lactation curve (-----).

$$Y = 13.366688 X^{-0.043835} e^{-0.002132 X} \quad R^2 = 0.98$$

2nd lactation - 2 milkings.

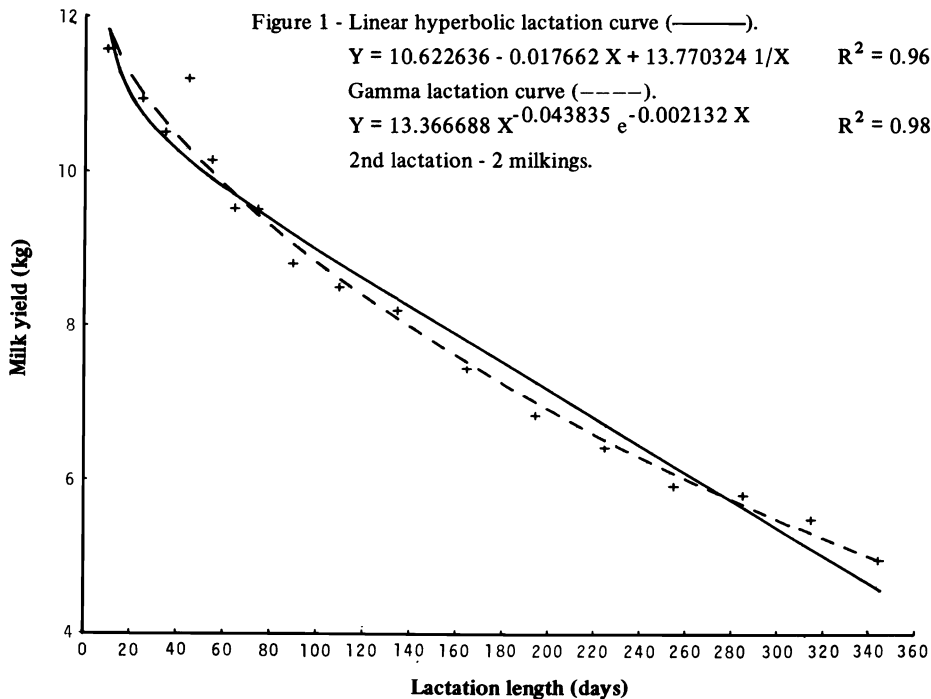


Figure 2 - Linear hyperbolic lactation curve (——).

$$Y = 11.357765 - 0.020823 X + 9.875588 1/X \quad R^2 = 0.96$$

Gamma lactation curve (-----).

$$Y = 13.124333 X^{-0.020194} e^{-0.002644 X} \quad R^2 = 0.96$$

3rd lactation - 2 milkings

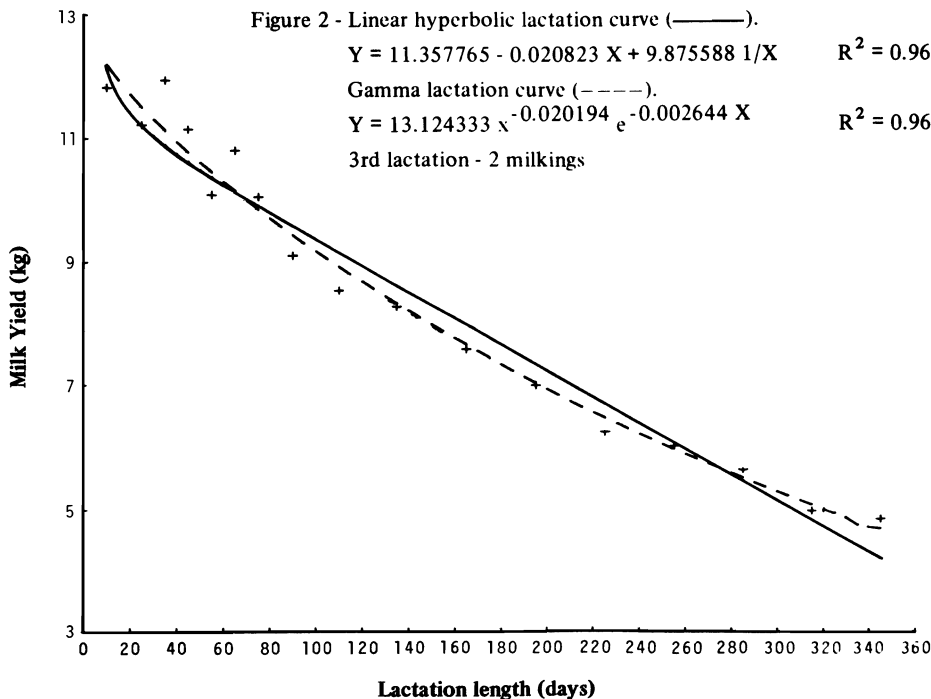


Table IV - Frequency distribution of the coefficients of determination R^2 .

Lactation	No. of milkings	Poor fit $R^2 < 0.50$						Regular fit $0.50 < R^2_A < 0.80$						Good fit $R^2 > 0.80$					
		L.H.		Gamma		L.H.		Gamma		L.H.		Gamma		L.H.		Gamma			
		No.	%	Animals	%	No.	%	Animals	%	No.	%	Animals	%	No.	%	Animals	%		
1st	2	130	31.0	133	31.1	181	43.2	178	42.5	108	25.5	108	25.8	25.8	108	25.8			
1st	3	8	17.8	9	20.0	12	26.7	16	35.6	25	55.5	25	44.4	20	44.4	44.4			
2nd	2	19	10.3	21	11.4	65	35.1	62	33.5	101	54.6	101	55.1	102	55.1	55.1			
2nd	3	13	6.8	14	7.3	51	26.7	59	30.9	127	66.6	127	61.8	118	61.8	61.8			
3rd	2	5	5.2	6	6.2	27	27.8	29	30.0	65	67.0	65	63.8	62	63.8	63.8			
3rd	3	6	4.1	6	4.1	23	15.6	30	20.4	118	80.3	118	75.5	111	75.5	75.5			

the linear hyperbolic curve than the gamma curve. If a good fit is considered $R^2 \geq 0.80$, the percentage of cows for the linear hyperbolic curve was lower only during the 2nd lactation with 2 milkings, i.e. 54.6% as opposed to 55.1% for the gamma curve. With a poor fit, the linear hyperbolic curve exhibited a lower percentage in all cases.

Also the number of cows with a good fit increased from one lactation to another of a higher order, with higher-level cows showing lower variability in production. This improvement in degree of fit also occurred with the gamma curve, suggesting that the distance between degrees decreased with herd selection. Furthermore, the lowest percentage of good fit (25.8%) and the highest percentage of poor fit (31.0%) occurred during the first lactation with 2 milkings, a fact that may have been due to absence of selection of the herd, whereas for cows milked three times these percentages were 55.5 and 17.8%, respectively.

Table V shows the analysis of variance for the estimated parameters of the linear hyperbolic curve. The shape of the linear hyperbolic lactation curve is highly influenced by order of lactation, calving season and number of milkings. The parameter β_1 (declining rate) was not affected by calving season, whereas the remaining parameters were. Figures 3 and 4 show the linear hyperbolic and gamma curves for first lactation, dry and rainy seasons, with 3 milkings.

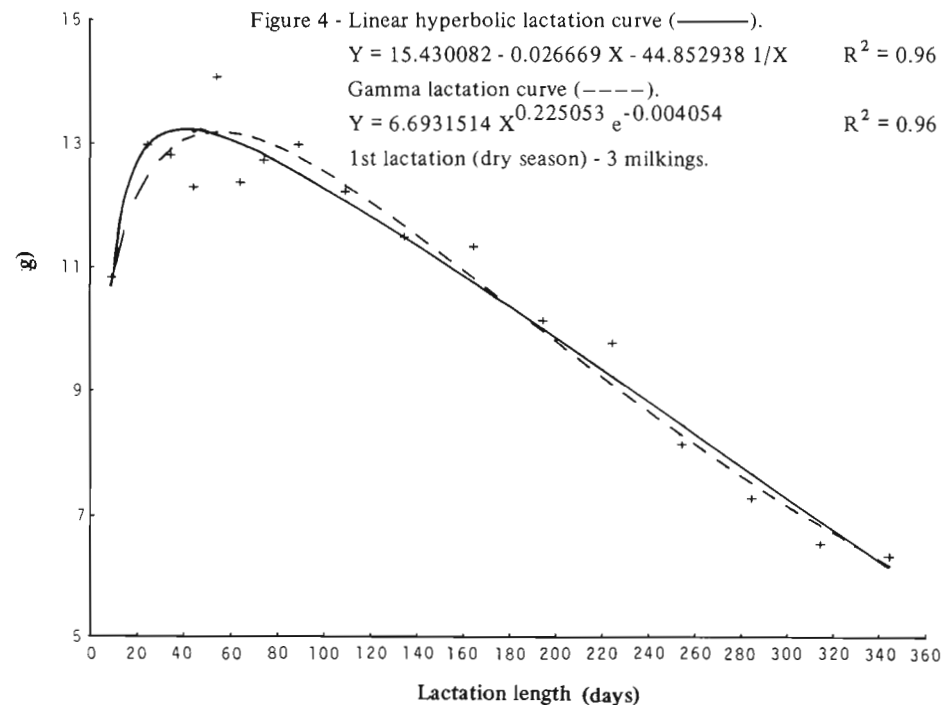
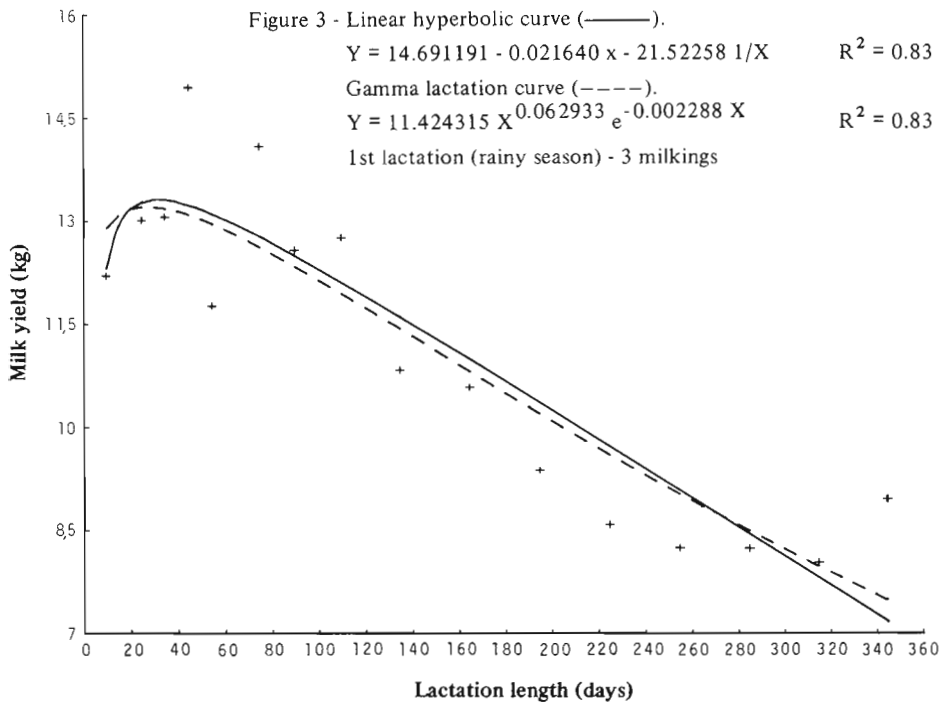
Table V - Analysis of variance of parameters estimated for the linear hyperbolic lactation curve.

Sources of variation	d.f.	β_0	β_1 ($\times 10^{-4}$)	β_2
Sire	26	15.54	1.99	5486.61
Cow/sire	469	12.18	1.64	5417.37
Lactation order	2	331.50***	4.27***	47185.25**
Calving season	1	123.62***	2.63	70330.11***
No. of milkings	1	1665.76***	109.52***	76508.67***
Error	583	7.96	1.27	5364.26

** Significant for $P < 0.01$; *** significant for $P < 0.001$.

CONCLUSIONS

Both the gamma and linear hyperbolic curves obtained for the means of the stages of lactation showed $R^2 \geq 0.96$, thus providing good fit. The lactation characteristics of Gyr cows show that gamma curves with negative b parameters may occur,



since these lactations do not always show growth from calving to peak and are practically linear. The linear hyperbolic curve showed slightly better fit than the gamma curve when both were used to describe lactations of a single animal. Parameter β_1 of the linear hyperbolic curve can be taken as the declining rate in kg of milk per day. Because of the simplicity of the equation, the linear hyperbolic curve can be used to advantage in relation to the gamma curve to describe lactations of Gyr cows, and also because in the present study 391 gamma curves were atypical, with a negative b parameter.

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RESUMO

Os modelos linear hiperbólico $Y = \beta_0 + \beta_1 X + \beta_2 1/X$ e gama $Y = Ax^b c^{-cx}$ foram utilizados para descrever a curva de lactação de 553 vacas da raça Gir exploradas para leite, considerando-se as três primeiras parições. Sobre as médias dos estágios da lactação, ambos modelos apresentaram um coeficiente de determinação $R^2 \geq 0,96$. Ao serem utilizados individualmente em todas as vacas do rebanho, mostrou-se que há uma ligeira superioridade de ajustamento do modelo linear hiperbólico sobre o gama. Devido à simplicidade de sua equação, o modelo linear hiperbólico pode ser usado com vantagens sobre o gama para a estimativa da produção total de uma vaca da raça Gir com registros incompletos.

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